# Unidad 4 : Método de la Transformada de Laplace

## Tema 4.3 : Teoremas de Traslación

| Primer Teorema de Traslación | Función Escalón Unitario |
|---|---|
| $f(t) \Rightarrow F(s)$<br>$e^{at}f(t) \Rightarrow F(s-a)$ | $U(t-a)=\begin{cases}0 & ; \quad 0\le t<a \\ 1 & ; \quad a\le t\le\infty\end{cases}$ |
| **Transformada del Escalón Unitario** | **Segundo Teorema de Traslación** |
| $U(t-a) \Rightarrow \frac{1}{s}e^{-as}$ | $f(t) \Rightarrow F(s)$<br>$f(t-a)U(t-a) \Rightarrow e^{-as}F(s)$ |
| **Función Impulso Unitario** | **Transformada del Impulso Unitario** |
| $\delta(t-a)=\frac{d}{dt}U(t-a)$ | $\delta(t-a) \Rightarrow e^{-as}$ |

### Ejemplos para la clase del 1er Teorema de Traslación:

| | |
|---|---|
| $E1: \quad L\{e^{at}t^n\}$ | $R1: \quad n!/(s-a)^{n+1}$ |
| $E2: \quad L\{e^{at}senbt\}$ | $R2: \quad b/\left[(s-a)^2+b^2\right]$ |
| $E3: \quad L\{e^{at}\cosh bt\}$ | $R3: \quad (s-a)/\left[(s-a)^2-b^2\right]$ |
| $E4: \quad L^{-1}\left\{\frac{s-5}{s^2+6s+11}\right\}$ | $R4: \quad e^{-3t}\cos(\sqrt{2}\,t)-\frac{8}{\sqrt{2}}e^{-3t}sen(\sqrt{2}\,t)$ |
| $E5: \quad L^{-1}\left\{\frac{5}{(s-1)^3}\right\}$ | $R5: \quad \frac{5}{2}t^2e^t$ |
| $E6: \quad L^{-1}\left\{\frac{1}{s^2+2s-8}\right\}$ | $R6: \quad \frac{1}{3}e^{-t}senh(3t)$ |

**Ejemplos para la clase del 2º Teorema de Traslación:**

| | |
|---|---|
| $E1: \quad L\left\{e^{3(t-2)}U(t-2)\right\}$ | $R1: \quad e^{-2s}/(s-3)$ |
| $E2: \quad L\left\{e^{3t}U(t-2)\right\}$ | $R2: \quad e^{-2(s-3)}/(s-3)$ |
| $E3: \quad L^{-1}\left\{\dfrac{e^{\frac{\pi}{2}s}}{s^2+9}\right\}$ | $R3: \quad \dfrac{1}{3}sen\left[3\left(t-\dfrac{\pi}{2}\right)\right]U\left(t-\dfrac{\pi}{2}\right) = \dfrac{1}{3}\cos(3t)U\left(t-\dfrac{\pi}{2}\right)$ |

**<u>Para la próxima clase estudiar las secciones:</u>**

| | | | | |
|---|---|---|---|---|
| 7.3 | Zill | 7.3 | Nagle | Teoremas de Traslación |
| 7.4 | Zill | 7.5 y 7.6 | Nagle | Otros Teoremas de Transformadas |

**<u>Tarea para entregar la próxima clase:</u>**

Tarea No. 23 : Teoremas de Traslación

## Primer Teorema de Traslación

**Primer Teorema de Traslación**

$$Si \quad L\{f(t)\} \quad = \quad F(s)$$

$$entonces$$

$$L\{e^{at} f(t)\} \quad = \quad F(s-a)$$

**Primer Teorema de Traslación**

$$Si \quad f(t) \quad \Rightarrow \quad F(s)$$

$$entonces$$

$$e^{at} f(t) \quad \Rightarrow \quad F(s-a)$$

**Demostración:**

$$Si \quad L\{f(t)\} = \int_0^\infty f(t) e^{-st} dt = F(s)$$

$$L\{e^{at} f(t)\} = \int_0^\infty e^{at} f(t) e^{-st} dt =$$

$$= \int_0^\infty f(t) e^{-(s-a)t} dt = F(s-a) \quad LCQD$$

**Ejemplo 1:**

$$Si \quad L\{t^n\} = \frac{n!}{s^{n+1}}$$

$$entonces$$

$$L\{e^{at} t^n\} = \frac{n!}{(s-a)^{n+1}}$$

**Ejemplo 2:**

$$Si \quad L\{\sin(bt)\} = \frac{b}{s^2 + b^2}$$

$$entonces$$

$$L\{e^{at} \sin(bt)\} = \frac{b}{(s-a)^2 + b^2}$$

**Ejemplo 3:**

$$Si \quad L\{\cos(bt)\} = \frac{s}{s^2 + b^2}$$

$$entonces$$

$$L\{e^{at} \cos(bt)\} = \frac{s-a}{(s-a)^2 + b^2}$$

**Ejemplo 4:**

$$Si \quad L\{\sinh(bt)\} = \frac{b}{s^2 - b^2}$$

$$entonces$$

$$L\{e^{at} \sinh(bt)\} = \frac{b}{(s-a)^2 - b^2}$$

**Ejemplo 5:**

$$Si \quad L\{\cosh(bt)\} = \frac{s}{s^2 - b^2}$$

$$entonces$$

$$L\{e^{at} \cos(bt)\} = \frac{s-a}{(s-a)^2 - b^2}$$

## Segundo Teorema de Traslación

**Segundo Teorema de Traslación**

$$Si \quad L\{f(t)\} = F(s)$$

$$entonces$$

$$L\{f(t-a)U(t-a)\} = e^{-as}F(s)$$

**Segundo Teorema de Traslación**

$$Si \quad f(t) \quad \Rightarrow \quad F(s)$$

$$entonces$$

$$f(t-a)U(t-a) \quad \Rightarrow \quad e^{-as}F(s)$$

**Demostración:**

$$Si \quad L\{f(t)\} = \int_0^\infty f(t)e^{-st}dt = F(s) \qquad entonces:$$

$$L\{f(t-a)U(t-a)\} = \int_0^\infty f(t-a)U(t-a)e^{-st}dt =$$

$$= \int_0^a f(t-a)0e^{-st}dt + \int_a^\infty \underbrace{f(t-a)e^{-st}}_{u=t-a}dt = 0 + \int_0^\infty f(u)e^{-s(u+a)}du$$

$$= e^{-as}\underbrace{\int_0^\infty f(u)e^{-su}du}_{F(s)} = \quad e^{-as}F(s) \quad L.C.Q.D.$$

**Ejemplo 1:**

$$Si \quad L\{e^{3t}\} = \frac{1}{s-3}$$

$$entonces$$

$$L\{e^{3(t-5)}U(t-5)\} = \frac{e^{-5s}}{(s-3)}$$

**Ejemplo 2:**

$$Si \quad L\{e^{3t}\} = \frac{1}{s-3}$$

$$entonces$$

$$L\{e^{3t}U(t-5)\} = L\{e^{3(t-5+5)}U(t-5)\} =$$

$$L\{e^{3(t-5)}e^{15}U(t-5)\} = e^{15}L\{e^{3(t-5)}U(t-5)\} =$$

$$e^{15}\frac{e^{-5s}}{(s-3)} = \frac{e^{-5s+15}}{(s-3)} = \frac{e^{-5(s-3)}}{(s-3)}$$

**Ejemplo 3:**

$$Si \quad L\{t\} = \frac{1}{s^2}$$

$$Si \quad L\{t^2\} = \frac{2}{s^3}$$

*entonces*

$$L\{(t-5)U(t-5)\} = e^{-5s}\frac{1}{s^2}$$

$$= \frac{e^{-5s}}{s^2}$$

$$L\{(t-5)^2U(t-5)\} = e^{-5s}\frac{2}{s^3}$$

$$= \frac{2e^{-5s}}{s^3}$$

**Ejemplo 4:**

$$Si \quad L\{t^2\} = \frac{2}{s^3}$$

*entonces*

$$L\{t^2U(t-5)\} = L\{(t-5+5)^2U(t-5)\} =$$

$$= L\{[(t-5)^2 + 10(t-5) + 25]U(t-5)\}$$

$$= L\{(t-5)^2U(t-5)\} + 10L\{(t-5)U(t-5)\}$$

$$+ 25L\{U(t-5)\}$$

$$= \frac{2e^{-5s}}{s^3} + 10\frac{e^{-5s}}{s^2} + 25\frac{e^{-5s}}{s}$$

**Ejemplo 5:**

$$Si \quad L^{-1}\{e^{-as}F(s)\} = f(t-a)U(t-a)$$

*entonces*

$$L^{-1}\left\{\frac{e^{-3s}}{s-7}\right\} = L^{-1}\left\{e^{-3s}\frac{1}{s-7}\right\}$$

$$= e^{7(t-3)}U(t-3)$$

**Ejemplo 6:**

$$Si \quad L^{-1}\{e^{-as}F(s)\} = f(t-a)U(t-a)$$

*entonces*

$$L^{-1}\left\{\frac{se^{-\frac{\pi}{2}s}}{s^2+36}\right\} = L^{-1}\left\{e^{-\frac{\pi}{2}s}\frac{s}{s^2+36}\right\}$$

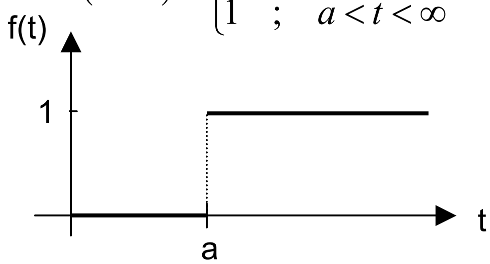
$$= cos\left[6\left(t-\frac{\pi}{2}\right)\right]U\left(t-\frac{\pi}{2}\right)$$

# Ecuaciones Diferenciales

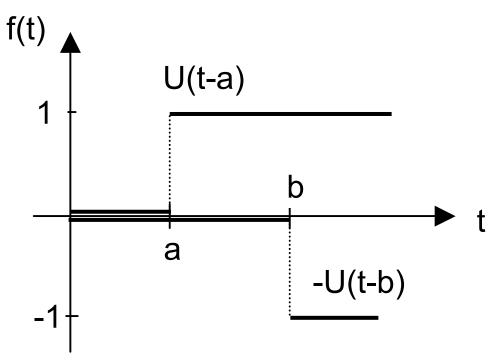
## Funciones Seccionalmente Continuas

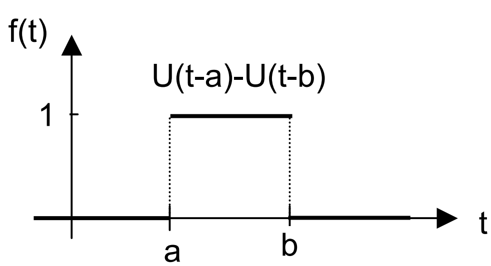
### Función Escalón Unitario

$$U(t-a)=\begin{cases}0 & ; \quad 0<t<a \\ 1 & ; \quad a<t<\infty\end{cases}$$

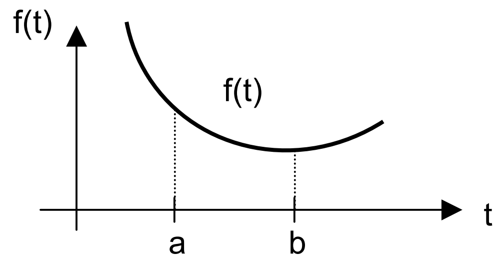


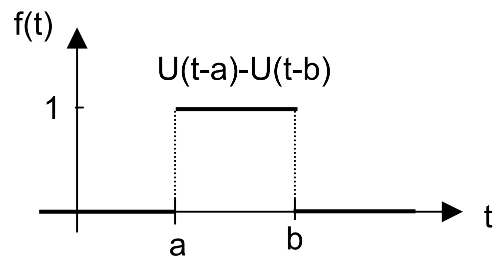

### Función Pulso Rectangular

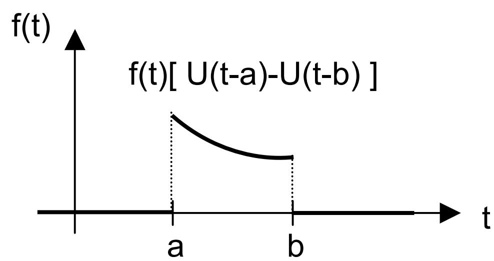



### *Obteniendo una sección de una función*



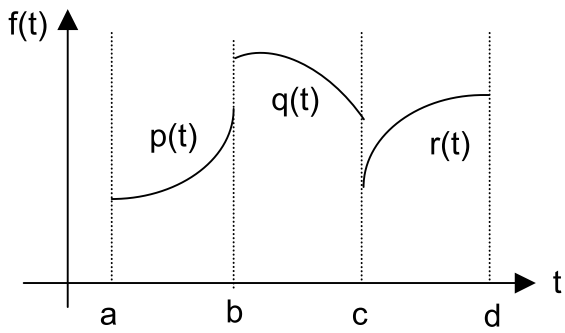




$$\delta(t-a)=0 \quad si \quad t \neq a \quad ; \quad \int_{-\infty}^{+\infty} \delta(t-a)\,dt = 1$$

$$\frac{d}{dt}U(t-a)=\delta(t-a)$$

### *Gráfica de una Función Seccionalmente Continua*



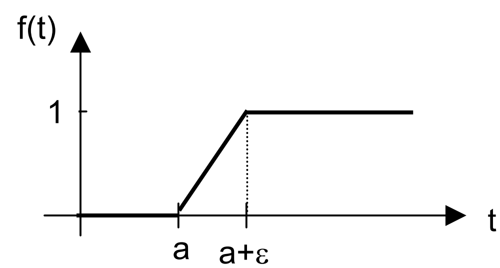

### *Ecuaciones de una Función Seccionalmente Continua*

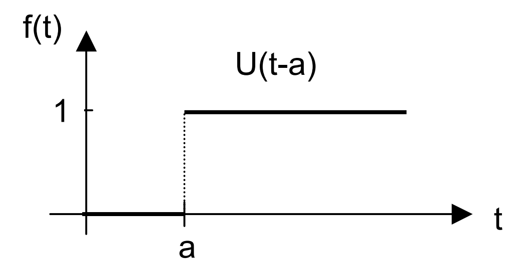
$$f(t)=\begin{cases} p(t) & ; \quad a<t<b \\ q(t) & ; \quad b<t<c \\ r(t) & ; \quad c<t<d \end{cases}$$

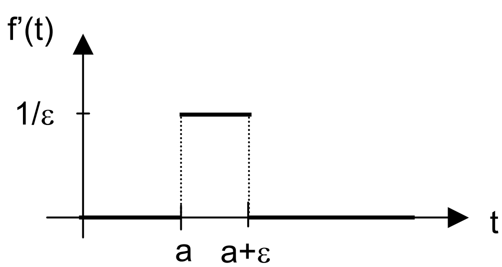
$$\begin{aligned} f(t) = {} & p(t)[U(t-a)-U(t-b)] \\ & + q(t)[U(t-b)-U(t-c)] \\ & + r(t)[U(t-c)-U(t-d)] \end{aligned}$$

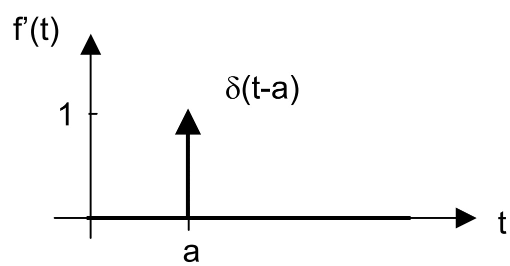
$$\begin{aligned} f(t) = {} & p(t)U(t-a) \\ & + [q(t)-p(t)]U(t-b) \\ & + [r(t)-q(t)]U(t-c) \\ & - r(t)U(t-d) \end{aligned}$$

### *Funciones Escalón e Impulso Unitario*




$$\int_{-\infty}^{+\infty} \delta(t-a)\,dt = 1$$

$$L\{\delta(t-a)\} = L\left\{\frac{d}{dt}U(t-a)\right\} = s\left(\frac{e^{-as}}{s}\right) - U(0) = e^{-as}$$

$$\underline{\underline{L\{\delta(t-a)\} = e^{-as}}} \quad ; \quad \underline{\underline{L\{\delta(t)\} = 1}}$$

## Ma-841 : Ecuaciones Diferenciales

### Tarea No. 23 : Teoremas de Traslación

En los siguientes problemas calcule las Transformadas o Transformadas Inversas indicadas.

$$P1:\quad L\left\{e^{5t}\,senh(3t)\right\}$$

$$P2:\quad L\left\{t\left(e^{t}+e^{2t}\right)^{2}\right\}$$

$$P3:\quad L^{-1}\left\{\frac{1}{s^{2}-6s+10}\right\}$$

$$P4:\quad L^{-1}\left\{\frac{s}{s^{2}+4s+5}\right\}$$

$$P5:\quad L^{-1}\left\{\frac{2s-1}{s^{2}(s+1)^{3}}\right\}$$

$$P6:\quad L\{tU(t-2)\}$$

$$P7:\quad L\{\cos(2t)U(t-\pi)\}$$

$$P8:\quad L\left\{(t-1)^{3}e^{t-1}U(t-1)\right\}$$

$$P9:\quad L^{-1}\left\{\frac{e^{-\pi s}}{s^{2}+1}\right\}$$

$$P10:\quad L^{-1}\left\{\frac{e^{-s}}{s(s+1)}\right\}$$

Respuestas

$$R1:\quad \frac{3}{(s-5)^{2}-9}$$

$$R2:\quad \frac{1}{(s-2)^{2}}+\frac{2}{(s-3)^{2}}+\frac{1}{(s-4)^{2}}$$

$$R3:\quad e^{3t}\,sent$$

$$R4:\quad e^{-2t}\cos t-2e^{-2t}\,sent$$

$$R5:\quad 5-t-5e^{-t}-4te^{-t}-\frac{3}{2}t^{2}e^{-t}$$

$$R6:\quad \frac{e^{-2s}}{s^{2}}+2\frac{e^{-2s}}{s}$$

$$R7:\quad \frac{s}{s^{2}+4}e^{-\pi s}$$

$$R8:\quad \frac{6e^{-s}}{(s-1)^{4}}$$

$$R9:\quad -\,sentU(t-\pi)$$

$$R10:\quad U(t-1)-e^{-(t-1)}U(t-1)$$